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STEINER TRIPLE SYSTEMS WITH ROTATIONAL AUTOMORPHISMS

Kevin T. PHELPS* and Alexander ROSA†

Department of Mathematical Sciences, McMaster University, Hamilton, Ontario, L8S 4K1, Canada

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A Steiner triple system $S(v)$ of order v is said to be *k-rotational* (k positive integer) if it admits an automorphism consisting of a single fixed point and precisely k $(v-1)/k$-cycles. We obtain a necessary and sufficient condition for the existence of 1-rotational and 2-rotational Steiner triple systems. We also enumerate nonisomorphic 1- and 2-rotational $S(v)$'s of small orders.

1. Introduction

A *Steiner system* $S(t, k, v)$ is a pair (V, B) where V is a v-set and B is a collection of k-subsets of V (called blocks) such that every t-subset of V is contained in exactly one block. Steiner systems $S(2, 3, v)$ are called *Steiner triple systems* (briefly STS) and will be denoted by $S(v)$; v is called the *order* of $S(v)$. It is well-known that an $S(v)$ exists if and only if $v \equiv 1$ or $3 \pmod 6$; orders satisfying this condition are admissible. Therefore whenever considering STSs it is understood that the considerations are restricted to admissible orders. An *automorphism* of an $S(v)$ (V, B) is a mapping of V onto itself which preserves B.

Let α be a permutation of degree v and type $[j] = [j_1, j_2, \ldots, j_v]$, i.e. α consists of precisely j_i cycles of length i, and $\sum ij_i = v$. An $S(v)$ admitting α as its automorphism will be denoted by $S_\alpha(v)$ or $S_{[j]}(v)$. One may ask the following question: given a permutation α of type $[j]$, what is the spectrum for $S_{[j]}(v)$'s, i.e. for what (admissible) orders v does there exist an $S_{[j]}(v)$? As far as we can tell this question has been answered completely (i.e. a nontrivial spectrum has been determined) only for two (nontrivial) types of permutations:

(i) $[j] = [0, \ldots, 0, 1]$, i.e. α consists of a single cycle of length v; any such $S_{[j]}(v)$ is called *cyclic*. It was shown first by Peltesohn [12] that a cyclic $S(v)$ exists if and only if $v \equiv 1$ or $3 \pmod 6$ and $v \neq 9$ (see also [11, 15, 17]).

(ii) $[j] = [1, \frac{1}{2}(v-1), 0, \ldots, 0]$, i.e. α is an involution fixing exactly one element; any such $S_{[j]}(v)$ is called *reverse*. It is known [9, 16, 18] that a reverse $S(v)$ exists if and only if $v \equiv 1$ or 3 or 9 or $19 \pmod{24}$.

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In this paper we consider the above question for permutations of type $[j^{(k)}] = [1, 0, \ldots, 0, k, 0, \ldots, 0]$ where k is a positive integer, $j_1 = 1$, $j_{(v-1)/k} = k$ and $j_i = 0$ otherwise; we call any $S_{[j^{(k)}]}(v)$ a *k-rotational $S(v)$*.

In Sections 2 and 3 we obtain necessary and sufficient conditions for the existence of 1-rotational and 2-rotational STSs, respectively. Section 4 discusses briefly k-rotational STSs for $k \geq 3$. Section 5 contains results concerning the (computer-assisted) enumeration of nonisomorphic 1-rotational and 2-rotational $S(v)$'s for $v \leq 27$ and $v < 25$, respectively.

2. The existence of 1-rotational Steiner triple systems

Let $Z = Z_{v-1} \cup \{\infty\}$, and let $\alpha = (\infty)(0\ 1 \cdots v-2)$ be an automorphism of a 1-rotational $S(v)$ (V, B). Since $\{\infty, i, j\} \in B$ implies $\{\infty, i+1, j+1\} \in B$, it follows that $\{\infty, i, j\} \in B$ if and only if $i - j \equiv \frac{1}{2}(v-1) \pmod{v-1}$; in other words, any 1-rotational $S(v)$ contains $\frac{1}{2}(v-1)$ triples of the form $\{\infty, i, i+\frac{1}{2}(v-1)\}$ mod v. All 3-subsets of V not containing the element ∞ are partitioned into orbits under α all of which are of length $v-1$ except possibly a single orbit Q_0 of length $\frac{1}{3}(v-1)$ of triples $\{0, \frac{1}{3}(v-1), \frac{2}{3}(v-1)\}$. It is easily seen that no 1-rotational $S(v)$ contains triples of Q_0: this would require $v \equiv 1 \pmod 6$, and at the same time, there would be need for further $\frac{1}{6}v\ (v-1) - \frac{1}{2}(v-1) - \frac{1}{3}(v-1) = \frac{1}{6}(v-1)(v-5)$ triples in B which would then necessarily have to be partitioned into $\frac{1}{6}(v-5)$ orbits of length $v-1$; this is obviously impossible as $\frac{1}{6}(v-5)$ is not an integer. Thus the remaining $\frac{1}{6}v(v-1) - \frac{1}{2}(v-1) = \frac{1}{6}(v-1)(v-3)$ triples of B fall into $\frac{1}{6}(v-3)$ orbits of length $v-1$. If $\{a, b, c\}$ is a triple in one such orbit then clearly the six differences $\pm(a-b)$, $\pm(a-c)$, $\pm(b-c)$ are all distinct, and if $\{a_1, b_1, c_1\}$, $\{a_2, b_2, c_2\}$ are two triples from two distinct orbits in B then the corresponding 12 differences are all distinct. Since there are still $v-3$ non-zero differences "available" it follows that $\frac{1}{6}(v-3)$ must be an integer, and so we must have

$$v \equiv 3 \pmod 6. \tag{1}$$

On the other hand, since v is odd, the automorphism $\alpha^{(v-1)/2}$ is a permutation of type $[j] = [1, \frac{1}{2}(v-1), 0, \ldots, 0]$, and so (V, B) is a reverse $S(v)$. It follows [16] that

$$v \equiv 1 \text{ or } 3 \text{ or } 9 \text{ or } 19 \pmod{24}. \tag{2}$$

The congruences (1) and (2) together yield

Lemma 2.1. *A necessary condition for the existence of a 1-rotational $S(v)$ is $v \equiv 3$ or 9 $\pmod{24}$.*

Before we proceed to the main result of this section, we need one more definition (which will be needed also in Section 3).

An *(A, k)-system* (a *(B, k)-system*, respectively) [15] is a set of ordered pairs

$\{(a_r, b_r) \mid r = 1, \ldots, k\}$ such that $b_r - a_r = r$ for $r = 1, \ldots, k$, and $\bigcup_{r=1}^{k} \{a_r, b_r\} = \{1, 2, \ldots, 2k\}$ $(= \{1, 2, \ldots, 2k-1, 2k+1\}$, respectively). An (A, k)-system and a (B, k)-system are essentially the same as a Skolem k-sequence and a hooked Skolem k-sequence, respectively [11, 17]. It is well-known that an (A, k)-system exists if and only if $k \equiv 0$ or 1 (mod 4), and a (B, k)-system exists if and only if $k \equiv 2$ or 3 (mod 4) (see [11, 15, 17]).

Theorem 2.2. *A* 1-*rotational* $S(v)$ *exists if and only if* $v \equiv 3$ *or* 9 (mod 24).

Proof. In view of Lemma 2.1, we have to demonstrate only the sufficiency of the condition. So let $v \equiv 3$ or 9 (mod 24), and let $V = Z_{v-1} \cup \{\infty\}$. Define a set B of triples on V as follows:

$$B = B_1 \cup B_2$$

where

$$B_1 = \{\{\infty, i, i + \tfrac{1}{2}(v-1)\} \mid i = 0, \ldots, \tfrac{1}{2}(v-3)\},$$

$$B_2 = \{\{i, i+r, i+b_r+k\} \mid i = 0, 1, \ldots, v-2;\ r = 1, \ldots, k\}$$

where $\{(a_r, b_r) \mid r = 1, \ldots, k\}$ is any (A, k)-system with $k = \frac{1}{6}(v-3)/6$; since $v \equiv 3$ or 9 (mod 24), $k \equiv 0$ or 1 (mod 4) and so an (A, k)-system exists. We claim that (V, B) is a 1-rotational $S(v)$. Indeed, given a pair of elements ∞, i where $i \in Z_{v-1}$, it is contained in exactly one triple of B_1 since clearly the set $\{\{i, i + \frac{1}{2}(v-1)\} \mid i = 0, 1, \ldots, \frac{1}{2}(v-3)\}$ partitions Z_{v-1}. Given a pair $i, j \in Z_{v-1}$, $i \neq j$, we look at their difference Δ_{ij}: if $\Delta_{ij} = \frac{1}{2}(v-1)$ then $\{i, j\}$ is contained in a unique triple of B_1. If $\Delta_{ij} = s \neq \frac{1}{2}(v-1)$ we may assume w.l.o.g. $1 \leqslant s \leqslant \frac{1}{2}(v-3)$. The six differences between the elements of a triple in B_2 are $\pm r$, $\pm(a_r + k)$, $\pm(b_r + k)$, and since the difference triples $\{\{r, a_r + k, b_r + k\} \mid r = 1, \ldots, k\}$ cover the set $\{1, 2, \ldots, \frac{1}{2}(v-3)\}$ it follows that the pair i, j with $\Delta_{ij} = s$ is contained in exactly one triple of B_2. Thus (V, B) is an $S(v)$. Observing that $\alpha = (\infty)(0\ 1 \cdots v-2)$ is an automorphism of (V, B) completes the proof.

3. The existence of 2-rotational Steiner triple systems

Let $V = (Z_{(v-1)/2} \times \{1, 2\}) \cup \{\infty\}$ and let

$$\alpha = (\infty)((0,1)(1,1) \cdots (\tfrac{1}{2}(v-1), 1))((0,2)(1,2) \cdots (\tfrac{1}{2}(v-1), 2))$$

be an automorphism of a 2-rotational $S(v)$. If $\frac{1}{2}(v-1) \equiv 0$ (mod 2) then $\alpha^{(v-1)/4}$ is a permutation of type $[j] = [1, \frac{1}{2}(v-1), 0, \ldots, 0]$ so that $S(v)$ is a reverse $S(v)$. But a reverse $S(v)$ cannot exist for $v \equiv 13$ or 21 (mod 24) thus we have

Lemma 3.1. *A necessary condition for the existence of a* 2-*rotational* $S(v)$ *is* $v \not\equiv 13$ *or* 21 (mod 24).

Before we proceed to determine the spectrum of 2-rotational STSs, we need a few auxiliary definitions and results.

Lemma 3.2. *If there exists a cyclic $S(v)$ then there exists a 2-rotational $S(2v+1)$.*

Proof. Let $V=Z_v$ and let (V, B) be a cyclic $S(v)$, with $\gamma=(0\ 1\cdots v-1)$ its cyclic automorphism. Put $W=(Z_v\times\{1,2\})\cup\{\infty\}$ where ∞ is a new symbol, and define a set of triples C on W as follows:

$$C=C_1\cup C_2\cup C_3$$

where

$$C_1=\{\{\infty,(i,1),(i,2)\}\mid i\in Z_v\},$$

$$C_2=\{\{(i,1),(i-j,2),(i+j,2)\}\mid i\in Z_v,\ j=1,2,\ldots,\tfrac{1}{2}(v-1)\},$$

$$C_3=\{\{(i,1),(j,1),(k,1)\}\mid\{i,j,k\}\in B\}.$$

It is a routine matter to see that (W, C) is an $S(2v+1)$, and that

$$\alpha=(\infty)((0,1)(1,1)\cdots(v-1,1))((0,2)(1,2)\cdots(v-1,2))$$

is its automorphism.

Lemma 3.3. *Any 1-rotational $S(v)$ is also 2-rotational.*

Proof. By Theorem 2.1, a 1-rotational $S(v)$ exists if and only if $v\equiv 3$ or $9\pmod{24}$. In this case $\frac{1}{2}(v-1)\equiv 0\pmod 2$, and so for any permutation α of type $[j]=[1,0,\ldots,0,1,0]$, α^2 is of type $[1,0,\ldots,0,2,0,\ldots,0]$ (i.e. $j_{(v-1)/2}=2$), and the statement of the lemma follows.

Definition 3.4. Let k be a natural number, and let

$$S(k)=\{1,2,\ldots,2k-1,2k+1,2k+2,\ldots,4k-1\},$$

$$T(k)=\begin{cases}\{2,3,4,\ldots,2k\} & \text{if } k \text{ is odd},\\ \{1,3,4,\ldots,2k\} & \text{if } k \text{ is even}.\end{cases}$$

A set of $2k-1$ ordered pairs $\{(c_r,d_r)\mid r\in T(k)\}$ such that $d_r-c_r=r$ for all $r\in T(k)$ and $\bigcup_{r\in T(k)}\{c_r,d_r\}=S(k)$ will be called an (F,k)-*system*.

Lemma 3.5. *An (F,k)-system exists if and only if $k\neq 2$.*

Proof. We have $T(2)=\{1,3,4\}$ but is easily seen that the set $S(2)=\{1,2,3,5,6,7\}$ cannot be partitioned into three pairs having differences $1,3,4$.

For $k=1,3,4,5$, an (F, k)-system is, e.g., as follows:

$$k=1:(1,3),$$

$$k=3:(1,3),\ (8,11),\ (5,9),\ (2,7),\ (4,10),$$

$$k=4:(13,14),\ (3,6),\ (11,15),\ (2,7),\ (4,10),\ (5,12),\ (1,9)$$

$$k=5:(6,8),\ (16,19),\ (14,18),\ (12,17),$$
$$(1,7),\ (2,9),\ (3,11),\ (4,13),\ (5,15).$$

Let now $k\geqslant 6$, and distinguish three cases:

Case 1. k even, $k=2s$, $k\geqslant 6$; then the following pairs form an (F, k)-system:

$$(r+1, 2k-r) \qquad r=1,\ldots,k-2,$$

$$(2k+1+r, 4k-1-r) \quad r=1,\ldots,s-2,$$

$$(5s-1+r, 7s-1-r) \quad r=1,\ldots,s-2,$$

$$(1,2k+1),\ (k,3k-2),\ (k+1,3k),\ (3k-1,4k-1),\ (7s-1,7s).$$

Case 2. $k\equiv 3 \pmod 4$, $k\geqslant 7$, $k=4s+3$; then the following pairs form an (F, k)-system:

$$(r+1, 2k-r) \qquad r=1,\ldots,k-2,$$

$$(2k+2r, 4k-2-2r) \qquad r=1,\ldots,2s,$$

$$(2k+1+2r, 4k+1-2r) \qquad r=1,\ldots,s,$$

$$(2k+2s+1+2r, 3k+2s-2r) \quad r=1,\ldots,s-1\ (s\geqslant 2),$$

$$(1,2k+1),\ (k,3k-2),\ (k+1,3k),\ (3k-1,4k-2),\ (3k+2s,3k+2s+2).$$

Case 3. $k\equiv 1 \pmod 4$, $k\geqslant 9$, $k=4s+1$; then the following pairs form an (F, k)-system:

$$(r, 2k-1-r) \quad r=1,\ldots,k-2,$$

$$(k-1,3k-2),\ (k,3k),\ (2k-1,4k-3),$$

and (a) for $s=2$

$$(19,29),\ (20,32),\ (21,35),\ (22,24),\ (23,31),\ (26,30),\ (28,34);$$

(b) for $s\geqslant 3$

$$(2k-1+2r, 4k-3-2r) \qquad r=1,\ldots,s,$$
$$(2k+2r, 4k-2r) \qquad r=1,\ldots,s,$$
$$(2k+2s+3+2r, 3k+2s-2-2r) \quad r=1,\ldots,s-3\ (s>3),$$
$$(2k+2s+2+4r, 3k+2s+3-4r) \quad r=1,\ldots,[\tfrac{1}{2}(s-1)]\ (s>2),$$
$$(2k+2s+4r, 3k+2s-3-4r) \qquad r=1,\ldots,[\tfrac{1}{2}(s-2)]\ (s>3),$$
$$(3k+2,4k-1),\ (2k+2s+1,2k+2s+3),$$

and

$$\begin{cases}(3k-3, 3k+1) & \text{if } s \text{ is odd,}\\ (3k-1, 3k+3) & \text{if } s \text{ is even.}\end{cases}$$

The verifications are tedious but straightforward. This completes the proof of the lemma.

Lemma 3.6. *If $v \equiv 1 \pmod{24}$, there exists a 2-rotational $S(v)$.*

Proof. The proof is by direct construction. Let

$$v = 24t+1, \qquad V = (Z_{12t} \times \{1, 2\} \cup \{\infty\},$$

$$\alpha = (\infty)(0_1\ 1_1 \cdots (12t-1)_1)(0_2\ 1_2 \cdots (12t-1)_2);$$

here instead of (x, i) we write for brevity x_i.

Define a set of triples B on V as follows:

$$B = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6$$

where the sets B_i are as follows (for brevity we write for each orbit under α only its representative; it is understood that the elements are developed modulo $12t$ to produce the remaining blocks of the orbit):

$$B_1 : \{\{\infty, 0_1, (6t)_1\}, \{\infty, 0_2, (6t)_2\}\},$$

$$B_2 : \{\{0_1, (4t)_1, (8t)_1\}\},$$

$$B_3 : \{\{0_1, r_1, (b_r-1)_2\} \mid r = 1, \ldots, 6t-1;\ 4 \neq 4t\}$$

where $\{(a_r, b_r) \mid r = 1, \ldots, 6t-1\}$ is an $(A, 6t-1)$-system or a $(B, 6t-1)$-system (see Section 2) depending on whether t is odd or even,

$$B_4 : \{\{0_1, (a_{4t}-1)_2, (b_{4t}-1)_2\}\},$$

$$B_5 : \begin{cases}\{\{0_2, 1_2, 2_1\}\} & \text{if } t \text{ is odd,}\\ \{\{0_2, 2_2, 3_1\}\} & \text{if } t \text{ is even,}\end{cases}$$

$$B_6 : \begin{cases}\{\{0_2, 1_2, 10_2\}, \{0_2, 5_2, 11_2\}, \{0_2, 3_2, 7_2\}\} & \text{if } t = 2,\\ \{\{0_2, (c_r+2t)_2, (d_r+2t)_2\} \mid r \in T(t)\} & \text{if } t \neq 2\end{cases}$$

where $\{(c_r, d_r) \mid r \in T(t)\}$ is any (F, t)-system (see Definition 3.4). Then (V, B) is an $S(v)$. In order to verify this claim, consider pairs of distinct elements of V. If such a pair contains the element ∞ then clearly it is contained in a unique triple of B_1. Suppose our pair is i_1, j_1. Then it is contained in a unique triple of B_1 if $i-j = 6t$, in a unique triple of B_2 if $i-j = \pm 4t$, and in a unique triple of B_3 otherwise (by virtue of how an (A, k)- or (B, k)-system was defined). Suppose our pair is i_1, j_2; then it is contained in a unique triple of B_4 if $j-i = a_{4t}-1$ or $b_{4t}-1$, in a unique triple of B_5 if $j-i = -1$ or -2 provided t is odd, and $j-1 = -1$ or -3 provided t is even, and in a unique triple of B_3 otherwise (if $t \neq 2$). Finally, if our pair is i_2, j_2

then it is contained in a unique triple of B_1 if $i-j=6t$, in a unique triple of B_4 if $i-j=\pm 4t$, in a unique triple of B_5 if $i-j=\pm 1$ provided t is odd, and if $i-j=\pm 2$ provided t is even, and in a unique triple of B_6 otherwise. Thus (V, B) is an $S(v)$; moreover, α is obviously an automorphism of (V, B).

Theorem 3.5. *A 2-rotational $S(v)$ exists if and only if $v \equiv 1$, 3, 7, 9, 15 or 19 (mod 24).*

Proof. The necessity was shown in Lemma 3.1. The sufficiency for orders $v \equiv 3$ or 9 (mod 24) follows from Theorem 2.2 and Lemma 3.3. Since a cyclic $S(v)$ exists for all orders $v \equiv 1$ or 3 (mod 6) except $v = 9$ [12] it follows by Lemma 3.2 that a 2-rotational $S(v)$ exists for all orders $v \equiv 3$ or 7 (mod 12) except possibly $v = 19$. However, a 2-rotational $S(19)$ exists, and, in fact, all such nonisomorphic systems are enumerated below in Section 5. Finally, the existence of a 2-rotational $S(v)$ for $v \equiv 1$ (mod 24) follows from Lemma 3.6.

4. k-rotational Steiner triple systems with $k \geq 3$

It can be shown, in a manner similar to that of Sections 2 and 3, that a necessary condition for the existence of a 3-rotational $S(v)$ is $v \equiv 1$ or 19 (mod 24), of a 4-rotational $S(v)$ is $v \equiv 1$, 9, 13 or 21 (mod 24), and of a 5-rotational $S(v)$ is $v \equiv 1, 51, 81$ or 91 (mod 120). The general existence problem for k-rotational STSs with $k = 3, 4, 5$ remains open although, for instance, it follows easily from Section 3 that 4-rotational $S(v)$ exists for all $v \equiv 1$ or 9 (mod 24), and a 5-rotational $S(v)$ exists for all $v \equiv 51$ or 81 (mod 120). Also, a 3-rotational $S(19)$ (see Example 1 below) and a 4-rotational $S(13)$ and $S(21)$ (see Example 2 below) do exist, and so it seems to us reasonable to conjecture that the above necessary conditions are also sufficient.

On the other hand, the following result is an easy corollary to the results of Sections 2 and 3.

Theorem 4.1. *A 6-rotational $S(v)$ exists if and only if $v \equiv 1$, 7 or 19 (mod 24).*

Example 1. A 3-rotational $S(19)$.

Elements: $V = (Z_6 \times \{1, 2, 3\}) \cup \{\infty\}$.

Base blocks:

$$\{\infty, 0_i, 3_i\} \quad i = 1, 2, 3,$$

$$\{0_i, 2_i, 4_i\} \quad i = 1, 2, 3,$$

$$\{0_1, 1_1, 2_2\}, \{0_2, 1_2, 4_3\}, \{0_3, 1_3, 3_1\}, \{0_1, 0_2, 0_3\},$$

$$\{0_1, 5_2, 1_3\}, \{0_1, 3_2, 2_3\}, \{0_1, 4_2, 5_3\}.$$

Example 2. A 4-rotational $S(21)$.

Elements: $V=(Z_5\times\{1,2,3,4\})\cup\{\infty\}$.

Base blocks:

$\{\infty,0_1,1_3\}$, $\{\infty,0_2,1_4\}$, $\{0_1,0_2,0_3\}$, $\{0_1,1_2,2_3\}$, $\{0_1,2_2,0_4\}$, $\{0_2,2_3,0_4\}$, $\{0_1,3_2,4_2\}$, $\{0_2,2_2,4_4\}$, $\{0_2,3_3,4_3\}$, $\{0_1,1_1,4_3\}$, $\{0_1,2_1,3_4\}$, $\{0_4,2_4,3_1\}$, $\{0_3,2_3,4_4\}$, $\{0_4,1_4,0_3\}$.

5. Enumeration of 1- and 2-rotational Steiner triple systems of small orders.

Denote by $N(v)$ the number of nonisomorphic $S(v)$'s. The values of $N(v)$ a known exactly for $v\leq 15$. However, $N(v)$ is known to grow very fast (asympto cally, $N(v)\sim\exp(\frac{1}{6}(v-1))$ so that the exact value of $N(v)$ for $v\geq 19$, besides bei hard to determine, is of little practical interest (one has already $N(19)>28400($ Therefore it seems more reasonable to strive for a (possibly constructive) enume ation of $S(v)$'s with certain additional properties so that the task becomes feasib and the obtained numerical values are "within reasonable bounds". Bays [2] w apparently first to move in this direction when he enumerated cyclic $S(v)$'s f $v\leq 37$ and $v=43$; recently, Colbourn [4] using computer, verified Bays's resul for $v\leq 37$ and extended the enumeration of cyclic $S(v)$'s up to $v\leq 45$, correcti in the process the value obtained by Bays for $v=43$ (see also [6]). The only oth result in this spirit, as far as we can tell, is due to Denniston [7], who has shov recently that there are exactly 184 nonisomorphic reverse $S(19)$'s.

We have determined that

(1) there are exactly 10 nonisomorphic 2-rotational $S(19)$'s, and

(2) there are exactly 35 nonisomorphic 1-rotational $S(27)$'s.

The systems were generated by hand but computer was used to test isomorp ism of obtained designs. Clique analysis was employed in the isomorphism testi in the former case while cycle structure was used to distinguish nonisomorph

Table 1. The ten 2-rotational Steiner triple systems of order 19

All systems contain the base blocks $\{\infty,0_1,0_2\}$, $\{0_1,3_1,6_1\}$ modulo 9, and also the following base blocks modulo 9:

No. 1	$0_2\,1_2\,3_2$	$5_1\,0_2\,4_2$	$3_1\,4_1\,0_2$	$6_1\,8_1\,0_2$	$2_1\,7_1\,0_2$
No. 2	$0_2\,1_2\,3_2$	$7_1\,0_2\,5_2$	$3_1\,4_1\,0_2$	$6_1\,8_1\,0_2$	$1_1\,5_1\,0_2$
No. 3	$0_2\,2_2\,3_2$	$5_1\,0_2\,4_2$	$3_1\,4_1\,2_2$	$6_1\,8_1\,0_2$	$2_1\,7_1\,0_2$
No. 4	$0_2\,2_2\,3_2$	$7_1\,0_2\,5_2$	$3_1\,4_1\,0_2$	$6_1\,8_1\,0_2$	$1_1\,5_1\,0_2$
No. 5	$0_1\,1_2\,3_2$	$5_1\,0_2\,4_2$	$6_1\,7_1\,0_2$	$2_1\,4_1\,0_2$	$3_1\,8_1\,0_2$
No. 6	$0_2\,1_2\,3_2$	$8_1\,0_2\,5_2$	$6_1\,7_1\,0_2$	$2_1\,4_1\,0_2$	$1_1\,5_1\,0_2$
No. 7	$0_2\,2_2\,3_2$	$5_1\,0_2\,4_2$	$6_1\,7_1\,0_2$	$2_1\,4_1\,0_2$	$3_1\,8_1\,0_2$
No. 8	$0_2\,2_2\,3_2$	$8_1\,0_2\,5_2$	$6_1\,7_1\,0_2$	$2_1\,4_1\,0_2$	$1_1\,5_1\,0_2$
No. 9	$0_2\,1_2\,3_2$	$6_1\,0_2\,4_2$	$1_1\,8_1\,0_2$	$4_1\,5_1\,0_2$	$3_1\,7_1\,0_2$
No. 10	$0_2\,1_2\,3_2$	$7_1\,0_2\,4_2$	$1_1\,8_1\,0_2$	$4_1\,5_1\,0_2$	$3_1\,7_1\,0_2$

Table 2. The thirty-five 1-rotational Steiner triple systems of order 27

All systems contain the base block $\{\infty, 0, 13\}$ modulo 26, and also the following four base blocks modulo 26:

No. 1	0 1 10	0 2 6	0 3 11	0 5 12
No. 2	0 1 10	0 2 6	0 8 11	0 5 12
No. 3	0 1 10	0 4 6	0 3 11	0 5 12
No. 4	0 1 10	0 4 6	0 8 11	0 5 12
No. 5	0 9 10	0 2 6	0 3 11	0 5 12
No. 6	0 9 10	0 2 6	0 8 11	0 5 12
No. 7	0 9 10	0 4 6	0 3 11	0 5 12
No. 8	0 9 10	0 4 6	0 8 11	0 5 12
No. 9	0 1 11	0 2 7	0 3 9	0 4 12
No. 10	0 1 11	0 2 7	0 6 9	0 4 12
No. 11	0 1 11	0 5 7	0 3 9	0 4 12
No. 12	0 1 11	0 5 7	0 6 9	0 4 12
No. 13	0 10 11	0 2 7	0 3 9	0 4 12
No. 14	0 10 11	0 2 7	0 6 9	0 4 12
No. 15	0 10 11	0 5 7	0 3 9	0 4 12
No. 16	0 10 11	0 5 7	0 6 9	0 4 12
No. 17	0 1 12	0 2 9	0 3 8	0 4 10
No. 18	0 1 12	0 2 9	0 5 8	0 4 10
No. 19	0 1 12	0 7 9	0 3 8	0 4 10
No. 20	0 1 12	0 7 9	0 5 8	0 4 10
No. 21	0 11 12	0 2 9	0 3 8	0 4 10
No. 22	0 11 12	0 2 9	0 5 8	0 4 10
No. 23	0 11 12	0 7 9	0 3 8	0 4 10
No. 24	0 11 12	0 7 9	0 5 8	0 4 10
No. 25	0 1 12	0 2 8	0 3 10	0 4 9
No. 26	0 1 12	0 2 8	0 3 10	0 5 9
No. 27	0 11 12	0 2 8	0 7 10	0 4 9
No. 28	0 1 5	0 2 8	0 3 14	0 7 16
No. 29	0 4 5	0 2 8	0 3 14	0 7 16
No. 30	0 1 5	0 2 8	0 11 14	0 7 16
No. 31	0 4 5	0 2 8	0 3 14	0 9 16
No. 32	0 1 11	0 2 8	0 3 7	0 5 14
No. 33	0 1 11	0 2 8	0 3 7	0 9 14
No. 34	0 1 11	0 2 8	0 4 7	0 9 14
No. 35	0 10 11	0 2 8	0 4 7	0 9 14

systems in the latter case (cf. [5]); in both cases, the respective invariants turned out to be complete.

The systems are listed in Tables 1 and 2. One of the 2-rotational $S(19)$'s is cyclic (system No. 8); it is the block-transitive design belonging to the class of designs of order $q \equiv 7 \pmod{12}$ described in [3] or [10]. The automorphism group of this system is of order 19·9. The remaining 9 systems have all automorphism group of order 9. For the sake of completeness, we remark that there are exactly three 2-rotational $S(15)$'s (see [20]).

Also, one of the 1-rotational $S(27)$'s (No. 31) is transitive; it is, in fact the design of points and lines of AG(3, 3).

Let us remark in conclusion that the existence and enumeration problem for designs with prescribed types of automorphisms has been considered for designs other than Steiner triple systems in the following cases: cyclic triple systems with $\lambda > 1$ [5], cyclic Steiner systems $S(2, k, v)$ for $k = 4, 5, 6$ [6], cyclic Steiner quadruple systems $S(3, 4, v)$ ([8], [14], to name only two), and rotational $S(3, 4, v)$ [13]. Some of the obtained enumeration results are remarkable, but apart from the case of cyclic triple systems with $\lambda > 1$, the existence problem in every other abovementioned case remains unsolved

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